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Heat transfer on a plate beneath an external uniform shear flow

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Abstract

The thermal characteristics of the flow over a semi-infinite flat plate driven by a uniform shear in the far field are investigated and compared to those of the corresponding classical Blasius flow problem. Similarity solutions are given in an exact analytic form in terms of the incomplete gamma function and the confluent hypergeometric function. Substantial differences are found concerning the scaling behavior of the wall heat flux for prescribed constant wall temperature T_w, as well as for the wall temperature distribution for prescribed constant heat flux q_w, both with respect to the wall coordinate x and the Prandtl number Pr. While for the Blasius flow different scaling laws hold for small and large values of Pr, in the uniform shear flow problem a universal scaling law is found for all Pr.

Keywords: Heat transfer; Boundary layer; Shear flow; Blasius flow; Prandtl number

1. Introduction

Shear driven flows, like the wall driven Couette flow, the wind-driven Eckman flow, the two-fluid parallel shear flow developed by Lock [1], etc. belong to the classical topics of fluid mechanics. Due to their wide breath of technical and environmental applications [2–4] the general research interest in shear driven flows [5–8] and their heat transfer characteristics [9,10] still remains a viable area of research.

The aim of the present investigation is to analyze the heat transfer characteristics on a semi-infinite flat plate due to an uniform shear flow $u = \beta y$, for prescribed conditions of wall temperature and heat flux compatible with a similarity analysis, and compare results with those available for the classical Blasius results for a uniform outer stream $u = U_\infty$. It is important to emphasize from the very beginning that this comparison concerns two fundamentally different physical flows. Indeed, whereas the classical Blasius flow is driven over the plate by an *inviscid* outer flow of *irrotational* velocity $U = U(x) = \text{const}$, our uniform shear flow is driven by a *viscous* outer flow of *rotational* velocity $U = U(y) = u(y) = \beta y$. Here β is the uniform strain rate and y is the plate normal coordinate; see Fig. 1. The uniform shear which drives the flow is $\tau = \mu(\mathrm{d}u/\mathrm{d}y) = \mu\beta = \text{const}$. Thus, one immediately recognizes that this uniform shear flow in fact is a Couette flow extended to the whole space. The plane surface which drives this extended Couette flow moves (in the mathematical model) with infinite velocity ($u = \beta y \to \infty$ as $y \to \infty$) at infinite distance from the fixed plate on which the temperature boundary layer is formed (and which is then compared to the temperature boundary

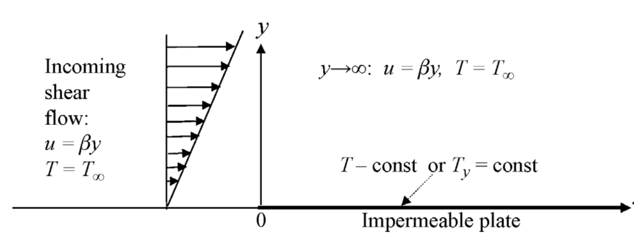

Fig. 1. Sketch of the plate, the coordinate system and the boundary conditions.

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Nomenclature

C_1, C_2	integration constants	β	strain rate
$f(\eta)$	similarity streamfunction variable	δ_T	edge of temperature boundary layer
$g(\eta)$	similarity temperature variable	$\gamma(a,z)$	incomplete gamma function
k	heat conductivity	$\Gamma(a)$	Gamma function
L	reference length	μ	dynamic viscosity
m	temperature power-law exponent	η	independent similarity variable
Nu	Nusselt number	θ	modified similarity temperature variable
Pr	Prandtl number	ν	kinematic viscosity
q	heat flux	ξ	modified independent similarity variable
t	dummy variable	ψ	streamfunction
T	temperature	*Subscripts*	
T_*	reference temperature	T	transverse
u, v	velocity components	w	wall conditions for all $x > 0$
x, y	Cartesian coordinates	∞	far field condition, $y \to \infty$
X	dimensionless wall coordinate	*Superscripts*	
Greek symbols		*prime*	derivative with respect to η or ξ
α	thermal diffusivity		

layer associated with the Blasius flow). In a real experimental set up, however, this $y = \infty$ means in fact (as it is interpreted usually in the boundary layer theory) the outer edge δ_T of the temperature boundary layer, such that the plate which drives our extended Couette flow moves (in the corresponding physical experiment) with the finite velocity $u = \beta\delta_T$.

2. Governing equations and boundary conditions

Consider planar shear flow over a semi-infinite insulated flat plate as sketched in Fig. 1. Using Cartesian coordinates (x, y) with corresponding velocities (u, v), the plate leading edge is fixed at $x = 0$ and we are interested in the temperature field T and streamwise velocity u for $x \geqslant 0$ when the exterior flow is $u = \beta y$, where β is a constant strain rate. Our basis of analysis is the Prandtl boundary layer equations for incompressible, zero pressure gradient flow

$$u_x + v_y = 0 \tag{1a}$$

$$uu_x + vu_y = \nu u_{yy} \tag{1b}$$

$$uT_x + vT_y = \alpha T_{yy} \tag{1c}$$

in which ν is the kinematic fluid viscosity and α is the thermal diffusivity, both assumed constant. Subscripts x and y denote partial derivatives with respect to those variables. The plate is impermeable and the no-slip boundary condition applies. We are interested in analyzing flows for which either the wall temperature is prescribed where one seeks for the wall heat transfer, or the opposite situation where the heat transfer is prescribed and one seeks the variation of the wall temperature. For both problems the boundary conditions are given by

$$u = 0 \tag{2a}$$

$$v = 0 \quad (y = 0,\ x \geqslant 0) \tag{2b}$$

with either a prescribed constant wall temperature or a constant wall heat flux

$$T(x, 0) = T_w \quad \text{or} \tag{2c}$$

$$q(x, 0) = -kT_y|_{y=0} = q_w \tag{2d}$$

and k is the fluid thermal conductivity.

In the far field the uniform shear and free stream temperature must be recovered and thus

$$u \to \beta y \tag{3a}$$

$$T \to T_\infty \quad (y \to \infty) \tag{3b}$$

Both problems of interest, given by boundary conditions (2c) and (2d), are embedded in a larger family of solutions whereby the wall temperature varies as a power-law in x, and hence we use this as our starting point for analysis in the following section.

3. Similarity transformation

Since the flow is incompressible and two-dimensional it is convenient to introduce a streamfunction $(u, v) = (\psi_y, -\psi_x)$ and write the governing equations and boundary conditions as

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = \nu\psi_{yyy} \tag{4a}$$

$$\psi_y = 0 \tag{4b}$$

$$\psi_x = 0 \quad (y = 0) \tag{4c}$$

$$\psi_y \to \beta y \quad (y \to \infty) \tag{4d}$$

$$\psi_y T_x - \psi_x T_y = \alpha T_{yy} \tag{5a}$$

$$T(x,0) = T_w \quad \text{or} \tag{5b}$$

$$q(x,0) = -kT_y|_{y=0} = q_w \tag{5c}$$

$$T \to T_\infty \quad (y \to \infty) \tag{5d}$$

A simple analysis shows (see, e.g. [2]) that the appropriate similarity variables for this problem with length scaled with $L = (\nu/\beta)^{1/2}$ and streamfunction scaled with ν may be posited in the form

$$\psi(x,y) = \nu\left(\frac{x}{L}\right)^{2/3} f(\eta) \tag{6a}$$

$$T(x,y) = T_\infty + T_*\left(\frac{x}{L}\right)^{m} g(\eta) \tag{6b}$$

$$\eta = \left(\frac{x}{L}\right)^{-1/3} \frac{y}{L} \tag{6c}$$

where T_* is a reference temperature to be specified below. Inserting (6a,c) into (4) yields the boundary-value problem for the flow variable

$$3f''' + 2ff'' - f'^2 = 0 \tag{7a}$$

$$f(0) = 0 \tag{7b}$$

$$f'(0) = 0 \tag{7c}$$

$$f'(\eta) \to \eta \quad (\eta \to \infty) \tag{7d}$$

and a coupled boundary-value problem for the similar temperature field

$$\frac{3}{Pr} g'' + 2fg' - 3mf'g = 0 \tag{8a}$$

$$g(0) = 0 \quad \text{or} \tag{8b}$$

$$g'(0) = -1 \tag{8c}$$

$$g(\eta) \to 0 \quad (\eta \to \infty) \tag{8d}$$

where $Pr = \nu/\kappa$ is the Prandtl number.

Clearly, $f(\eta) = \eta^2/2$ satisfies both Eq. (7a) and its attendant boundary conditions (7b–d) and thus represents a solution uniformly valid for all η. Inserting $f(\eta)$ and its derivatives into Eq. (8a) yields the similarity equation for the temperature field, namely

$$\frac{3}{Pr} g'' + \eta^2 g' - 3m\eta g = 0 \tag{9}$$

In this formulation the dimensional wall heat flux is given by

$$q_w(x) = -\frac{kT_*}{L}\left(\frac{x}{L}\right)^{m-1/3} g'(0) \tag{10}$$

Thus the exponents of interest are $m = 0$ for a prescribed constant wall temperature, and $m = 1/3$ for a prescribed constant wall heat flux. These cases will be considered separately in the following sections.

4. Constant wall temperature

Setting $m = 0$ in governing equation (9) and using boundary conditions (8b,d), we obtain the boundary-value problem corresponding to shear flow over a flat plate with uniform wall temperature

$$\frac{3}{Pr} g'' + \eta^2 g' = 0 \tag{11a}$$

$$g(0) = 1 \tag{11b}$$

$$g(\infty) = 0 \tag{11c}$$

Separating variables in (11a) and performing the necessary quadratures satisfying conditions (11b,c) yields immediately the solution

$$g(\eta) = 1 - \frac{1}{\Gamma(1/3)} \int_0^z t^{-2/3} e^{-t}\, dt = 1 - \frac{\gamma(1/3, z)}{\Gamma(1/3)} \tag{12a}$$

where

$$z = \frac{Pr}{9}\eta^3 \tag{12b}$$

and $\gamma(a,z)$ is the incomplete gamma function (cf. [11, Chapter 6], [12, Chapter 8]).

Evaluation of the derivative of (12a) readily gives

$$-g'(0) = \frac{(3Pr)^{1/3}}{\Gamma(1/3)} = 0.538366\, Pr^{1/3} \tag{13}$$

In Eq. (10) we have in this case $T_* = T_w - T_\infty$ and thus arrive at the dimensional wall heat flux

$$q_w(x) = \frac{k(T_w - T_\infty)}{L} \frac{(3Pr)^{1/3}}{\Gamma(1/3)} \left(\frac{x}{L}\right)^{-1/3} \tag{14}$$

Similar temperature profiles $g(\eta)$ at the selected values $Pr = 0.1, 0.3, 1$ and 5 are displayed in Fig. 2.

It is interesting to note that the present boundary-value problem (11) is encountered in several contexts in fluid mechanics in the same or in a slightly rescaled form. Thus it

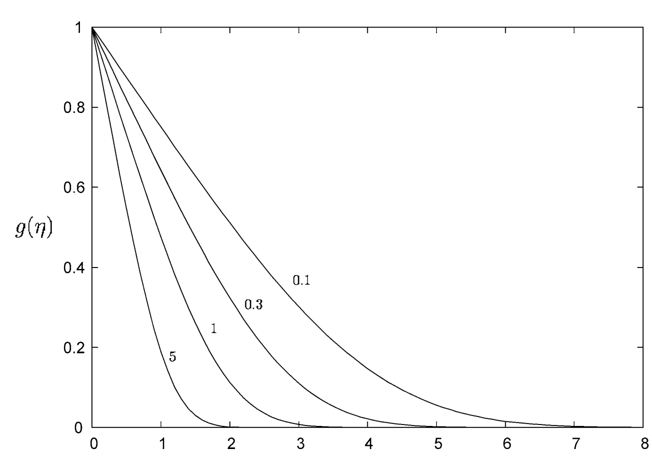

Fig. 2. Similar temperature profiles $g(\eta)$ according to Eq. (12a) for selected values of the Prandtl number (as indicated) when a constant wall temperature T_w is prescribed.

also arises in a three-dimensional boundary layer flow analysis, as reported by Weidman [6]. In that situation the problem is to determine, for a given streamwise uniform shear flow of strain rate β, possible fully-developed cross flows that may exist. One transverse flow is a different uniform shear flow of strain rate β_T. However, a second cross flow exists which is exactly that given by solution (12) above, but with Pr replaced by β/ν. Furthermore, the boundary-value problem (11) also arises in connection with the large Prandtl number ($Pr \to \infty$) heat transfer in different wall bounded flows (see Schlichting and Gersten [2, Section 9.3]). These boundary layer flows can be driven over the flat plate with prescribed temperature distribution $T_w = T_w(x)$ either by an outer inviscid (potential) flow of velocity $U = U(x)$ (Blasius flow, stagnation point flow, etc.) or, they are momentum driven submerged flows (heated wall jets) in a quiescent viscous fluid [2].

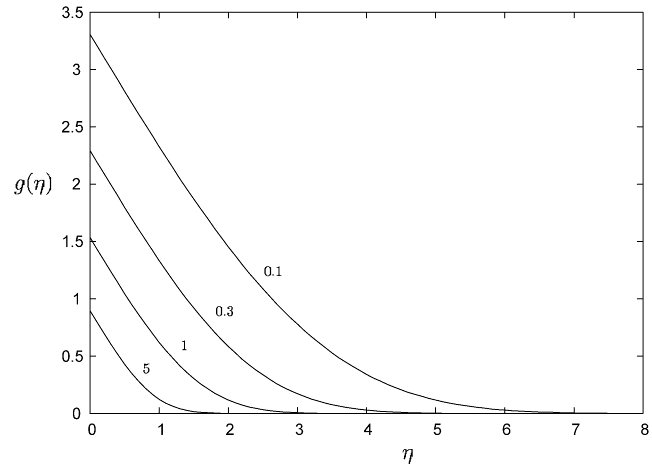

Fig. 3. Similar temperature profiles $g(\eta)$ according to Eq. (21) for selected values of the Prandtl number (as indicated) when a constant wall heat flux q_w is prescribed.

5. Constant heat flux

Now setting $m = 1/3$ in governing equation (9) and using boundary conditions (8c,d), we obtain the boundary-value problem corresponding to shear flow over a flat plate with uniform wall heat flux

$$\frac{3}{Pr} g'' + \eta^2 g' - \eta g = 0 \tag{15a}$$

$$g'(0) = -1 \tag{15b}$$

$$g(\infty) = 0 \tag{15c}$$

The Prandtl number may be eliminated through the scaling

$$g(\eta) = \left(\frac{3}{Pr}\right)^{1/3} \theta(\xi) \tag{16a}$$

$$\eta = \left(\frac{3}{Pr}\right)^{1/3} \xi \tag{16b}$$

to obtain

$$\theta'' + \xi^2 \theta' - \xi\theta = 0 \tag{17a}$$

$$\theta'(0) = -1 \tag{17b}$$

$$\theta(\infty) = 0 \tag{17c}$$

Finally, the change of independent variable $\xi = -(3t)^{1/3}$ furnishes the boundary value problem

$$t\frac{d^2\theta}{dt^2} + \left(\frac{2}{3} - t\right)\frac{d\theta}{dt} + \frac{1}{3}\theta = 0 \tag{18a}$$

$$\lim_{t\to 0}\left\{(3t)^{2/3}\frac{d\theta}{dt}\right\} = 1 \tag{18b}$$

$$\lim_{t\to -\infty} \theta = 0 \tag{18c}$$

which is the canonical form for identifying the general solutions in terms of confluent hypergeometric functions (see [11, Chapter 13], [12, Chapter 9])

$$\theta(\xi) = C_1 M\left(-\frac{1}{3}, \frac{2}{3}, t\right) + C_2 t^{1/3} \tag{19}$$

Satisfying the limit condition (18b) gives C_2, and application of the far field condition (18c) gives C_1, namely

$$C_1 = \frac{3^{1/3}}{\Gamma(2/3)} \tag{20a}$$

$$C_2 = 3^{1/3} \tag{20b}$$

Reverting back to the original similarity variables, one obtains the desired solution

$$g(\eta) = -\eta + \frac{3^{2/3}}{\Gamma(2/3)} Pr^{-1/3} M\left(-\frac{1}{3}, \frac{2}{3}, -\frac{Pr}{9}\eta^3\right) \tag{21}$$

From this one finds the similar wall temperature is equal to

$$g(0) = \frac{3^{2/3}}{\Gamma(2/3)} Pr^{-1/3} = 1.536117\, Pr^{-1/3} \tag{22}$$

Sample temperature profiles $g(\eta)$ at the selected values of Pr are presented in Fig. 3. In Eq. (10) we now have $T_* = q_w L/k$ where q_w is the specified wall heat flux. Thus the corresponding dimensional temperature field is

$$T(x, y) = T_\infty + \frac{q_w L}{k}\left(\frac{x}{L}\right)^{1/3} g(\eta) \tag{23}$$

with $g(\eta)$ given by Eq. (21). For an imposed constant heat flux, the plate temperature varies as

$$T_w(x) = T_\infty + \frac{q_w L}{k}\frac{3^{2/3}}{\Gamma(2/3)} Pr^{-1/3}\left(\frac{x}{L}\right)^{1/3} \tag{24}$$

6. Discussion and conclusions

The thermal characteristics of Blasius flow over a hot (or cold) flat plate are well known (see Schlichting and Gersten [2, Section 9.3]) and are summarized for $T_w =$ const and $q_w =$ const in Table 1 below. At the same downstream location, these classical results are compared in Table 1 with the

Table 1
Comparison of the thermal characteristics of the uniform shear flow and Blasius flow for constant wall temperature and heat flux, respectively

	Uniform shear flow (present paper) $L_{\text{shear}} = (\nu/\beta)^{1/2}$	Blasius flow (Ref. [2]) $L_{\text{Blasius}} = \nu/U_\infty$
Constant temperature boundary conditions: $T_w = \text{const}$		
$Nu = \dfrac{q_w(x)L}{k(T_w - T_\infty)} =$	$\dfrac{3^{1/3}}{\Gamma(1/3)} Pr^{1/3} \left(\dfrac{x}{L}\right)^{-1/3}$	$-g'(0)\left(\dfrac{x}{L}\right)^{-1/2}$; $-g'(0) = \begin{cases} \sqrt{\frac{2}{\pi}} Pr^{1/2}, & Pr \to 0 \\ 0.479 Pr^{1/3}, & Pr \to \infty \end{cases}$
Constant heat flux boundary conditions: $q_w = \text{const}$		
$\dfrac{1}{Nu} = \dfrac{k(T_w(x) - T_\infty)}{q_w L} =$	$\dfrac{3^{2/3}}{\Gamma(2/3)} Pr^{-1/3} \left(\dfrac{x}{L}\right)^{1/3}$	$g(0)\left(\dfrac{x}{L}\right)^{1/2}$; $g(0) = \begin{cases} \sqrt{\frac{2}{\pi}} Pr^{-1/2}, & Pr \to 0 \\ 1.524 Pr^{-1/3}, & Pr \to \infty \end{cases}$

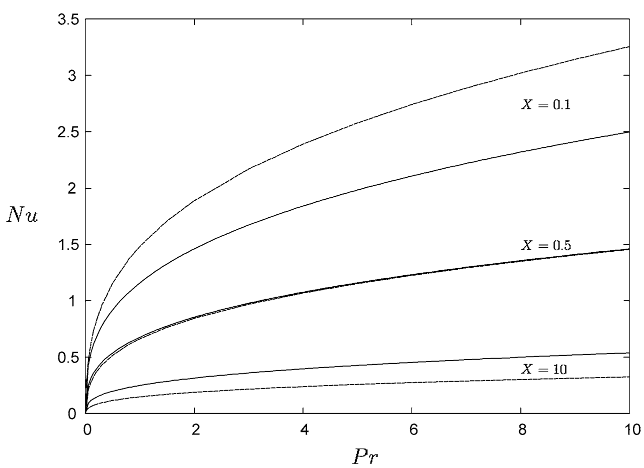

Fig. 4. Comparison of the Prandtl number variation of Nusselt number for uniform shear boundary layer flow (solid lines) and Blasius flow (dashed lines) with prescribed constant wall temperature T_w at three downstream locations $X = 0.1, 0.5$ and 10 when the dimensionless length scales for the two problems are equal: $L_{\text{Blasius}} = L_{\text{shear}}$ as explained in the text.

corresponding thermal characteristics of the uniform shear flow presented in Sections 4 and 5 above.

It is important to emphasize again that Table 1 compares the heat transfer characteristics of fundamentally different physical flows. Indeed, whereas the classical Blasius flow is driven over the plate by an *inviscid* outer flow of *irrotational* velocity $U = U(x) = \text{const}$, our uniform shear flow is driven by a *viscous* outer flow of *rotational* velocity $U = U(y) = \beta y$. The only common physical feature of these two flows is that neither of them possesses a natural characteristic length. Hence, it does not make sense to compare these two flows with each other, unless one compares them on the same length scale L. The choice of the length scale is required in the present context by the definition of the Nusselt number (see Table 1). The reference length chosen for the Blasius flow and that already chosen for our shear driven flow are $L_{\text{Blasius}} = \nu/U_\infty$ and $L_{\text{shear}} = \sqrt{\nu/\beta}$, respectively. In this way, the requirement $L_{\text{Blasius}} = L_{\text{shear}}$ implies $\beta = U_\infty^2/\nu$. Therefore, the Blasius flow is compared in Table 1 with the shear flow of velocity $u = \beta y = (U_\infty^2/\nu)y$.

For constant prescribed wall temperature T_w, the suitable dimensionless quantity for a physical comparison is the Nusselt number *Nu*, while for constant prescribed wall heat flux q_w the physically relevant quantity is the wall temperature distribution or, in a dimensionless form, the reciprocal Nusselt number (see Table 1). In both cases essential differences occur in the x-dependence as well as in the *Pr* scaling of *Nu* and $1/Nu$, respectively. While in the Blasius case *Nu* and $1/Nu$ scale differently for small and large values of *Pr*, in the case of uniform shear a universal scaling law, $Pr^{1/3}$ and $Pr^{-1/3}$, respectively, is found. This is made clear in Fig. 4 where *Nu* is plotted as a function of *Pr* for three downstream positions $X = x/L$ for both the Blasius and shear flow in the case $T_w = \text{const}$ (Table 1). Note that near the leading edge at $X = 0.1$ the Blasius heat transfer is greater than that for the shear flow, while the reverse is true far downstream at $X = 10$. At the intermediate value $X = 0.5$ the Nusselt numbers for the two flows are nearly equal for all vales of the Prandtl number. However, a small difference appears at low *Pr* as it must, owing to the different Prandtl number scalings $Pr^{-1/3}$ for the uniform shear flow and $Pr^{-1/2}$ for the Blasius flow.

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